SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2009
YEAR 12 Mathematics Extension 1
HSC Task #3

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each Section is to be returned in a separate bundle.
- All answers must be given in exact simplified form unless otherwise stated.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.

Total Marks – 69

- Attempt questions 1-3
- Start each new section in a separate answer booklet

Examiner: *D.McQuillan*

Question 1 (24 marks)

Marks

(a) Differentiate

(i) e^{2x} **1**

(ii) $\ln(2x-5)$ **1**

(iii) x^2e^{4x} **2**

(iv) $\dfrac{\ln x}{x}$ **2**

(b) Find

(i) $\int e^{1-x}\,dx$ **1**

(ii) $\int \dfrac{dx}{2x-7}$ **1**

(c) Evaluate

(i) $\int_0^1 e^{4x}\,dx$ **2**

(ii) $\int_3^7 \dfrac{3x}{x^2-5}\,dx$ **2**

(d) Evaluate

(i) $\tan^{-1}\left(\sqrt{3}\right)$ **1**

(ii) $\sin^{-1}\left(\sin\dfrac{\pi}{4}\right)$ **1**

(iii) $\cos\left(2\sin^{-1}\left(\dfrac{5}{13}\right)\right)$ **2**

(e) Using the substitution $u = x^2$ find $\int 2xe^{x^2}\,dx$. **2**

(f) Find the equation of the tangent to the curve $y = e^{3x-1}$ at the point where $x = 2$. **3**

(g) Solve the following pair of equations simultaneously. **3**

$$\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$$

$$\sin^{-1} x - \sin^{-1} y = \frac{\pi}{6}$$

End of Question 1

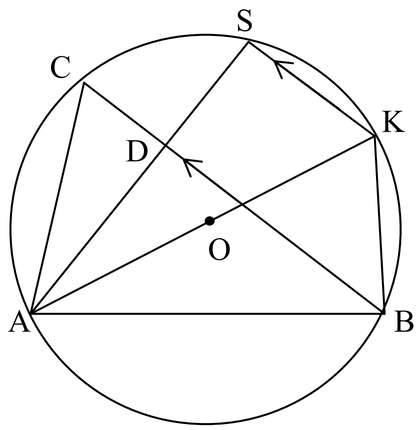
START a new ANSWER BOOKLET

Question 2 (24 marks) **Marks**

(a) AK is a diameter of the circle, centre O. SK||CB.

(i) Prove that $AS \perp BC$. **2**

(ii) Let $\angle BAK = \alpha$, hence show that $\angle SAC = \alpha$. **2**



(b)

(i) Show that $u^2 + u + 1 + \dfrac{1}{u-1} = \dfrac{u^3}{u-1}$. **1**

(ii) Hence find $\displaystyle\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}}$ using the substitution $x = u^6$. **2**

(c) The curve $y = \dfrac{3}{\sqrt{x^2+4}}$ is rotated about the x-axis between $x = 0$ and $x = 2$. Find the volume of the solid generated. **3**

(d) Given that $f(x) = 2 - \dfrac{1}{2e^{x^3}}$

(i) show that $f'(x) + 3x^2 f(x) = 6x^2$ **2**

(ii) show that the continuous function $f(x)$ has a root between $x = -2$ and $x = -1$ **1**

(iii) use one application of Newton's Method with initial approximation of $x = -1$ to find an approximation to the root of $f(x)$. Give your answer to 3 decimal places. **2**

(e) Evaluate $\int_{-3}^{0} \frac{x^2}{\sqrt{1-x}}dx$ using the substitution $u = 1 - x$. **3**

(f) Let $y = \cot^{-1} x$ be defined as $x = \cot y$ for $0 < y < \frac{\pi}{2}$.

(i) Graph $y = \cot^{-1} x$. **1**

(ii) Show that $\frac{dy}{dx} = -\frac{1}{1+x^2}$. **2**

(iii) Hence or otherwise show that, for all $x > 0$,

$$f(x) = \tan^{-1} x + \cot^{-1} x$$

is a constant function. **2**

(iv) Find the value of $\tan^{-1} x + \cot^{-1} x$ in exact form. **1**

End of Question 2

START a new ANSWER BOOKLET

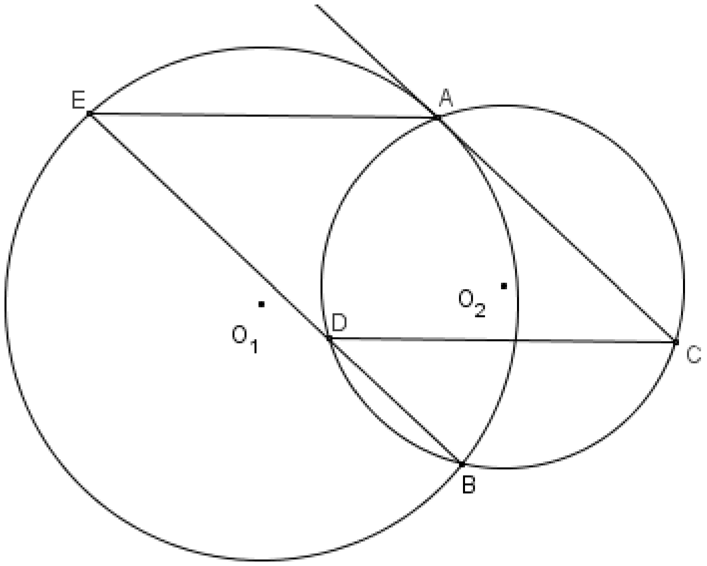
Question 3 (21 marks)

Marks

(a) Let $I = \int_o^{\pi} x f(\sin x) dx$. Using the substitution $u = \pi - x$ show that

$2I = \pi \int_o^{\pi} f(\sin x) dx$. **3**

(b) Two circles O_1 and O_2 meet at the points A and B. When produced, the tangent to O_1 at A meets O_2 at C. A point E lies on the circumference of O_1 so that BE is parallel to CA. The chord BE meets O_2 at D. Prove that AE = CD. **3**



(c) Using Mathematical Induction, prove that for any integer $n \geq 1$,

$$5 \times 2^{3n-2} + 3^{3n-1}$$

is divisible by 19. **3**

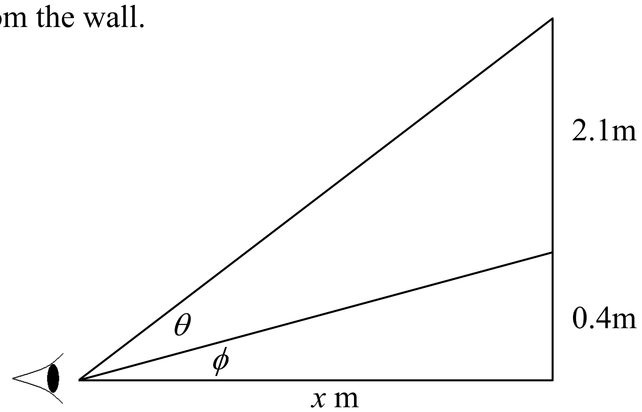
(d) Given that $f(x) = \dfrac{e^x - e^{-x}}{2}$.

(i) Show that $f^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ **3**

(ii) Solve the equation $f(x) = 5$ to 2 decimal places. **1**

(iii) Find $\dfrac{d}{dx}\left[f^{-1}(x)\right]$. **2**

(e) A painting in an art gallery has height 2.1 m and is hung so that its lower edge is 0.4 metres above the eye of an observer who is standing x metres away from the wall.



(i) Find an expression for ϕ in terms of x. **1**

(ii) Hence, find θ as a function of x. **2**

(iii) How far from the wall should the observer stand to get the best view? (That is, find x such as to maximise the viewing angle θ.) **3**

End of Question 3

End of Exam

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

QUESTION 1

(a) (i) $2e^{2x}$.

(ii) $\frac{2}{2x-5}$

(iii) $2xe^{4x} + 4x^2e^{4x}$

(iv) $\left(\frac{1}{x}\ln x\right)' = -x^{-2}\ln x + \frac{1}{x^2}$

$= \frac{1}{x^2} - \frac{\ln x}{x^2}$.

(b) (i) $\int e^{1-x}dx = -e^{1-x} + C.$

(ii) $\int \frac{dx}{2x-7} = \frac{1}{2}\int \frac{2dx}{2x-7}$

$= \frac{1}{2}\ln(2x-7) + C.$

(c) (i) $\int_0^1 e^{4x}dx = \frac{1}{4}e^{4x}\Big|_{x=0}^{1}$

$= \frac{e^4}{4} - \frac{1}{4}.$

(ii) $\int_3^7 \frac{3x}{x^2-5}\,dx = \frac{3}{2}\int_3^7 \frac{2x}{x^2-5}\,dx$

$= \frac{3}{2}\left[\ln(x^2-5)\right]_3^7$

$= \frac{3}{2}\left(\ln 44 - \ln 4\right).$

$= \frac{3}{2}\ln 11$

(d) (i) $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.

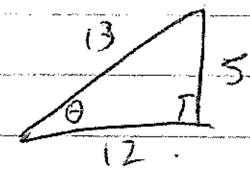
(ii) $\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4}$.

(iii) $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right) = \cos(2\theta)$.

$= \cos^2\theta - \sin^2\theta$.

$= \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2$.



$= \frac{144-25}{169}$

$= \frac{119}{169}$

(e) $\int 2xe^{x^2}\,dx$ $\qquad u = x^2$

$= \int e^u\,du$. $\qquad \frac{du}{dx} = 2x$.

$du = 2x\,dx$.

$= e^u + c$

$= e^{x^2} + C$.

(f) $y = e^{3x-1}$ $\qquad x = 2$.

$\frac{dy}{dx} = 3e^{3x-1}$.

Grad at $x=2$. $\qquad (2, e^5)$.

$m = 3e^5$.

$y - e^5 = 3e^5(x-2)$.

$y = 3e^5x - 5e^5$.

(9) $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$ (A)

$\sin^{-1}x - \sin^{-1}y = \frac{\pi}{6}$ (B)

(A) + (B)

$2\sin^{-1}x = \frac{2\pi}{3}$

$\sin^{-1}x = \frac{\pi}{3}$

$x = \frac{\sqrt{3}}{2}$

(A) − (B).

$2\sin^{-1}y = \frac{\pi}{3}$

$\sin^{-1}y = \frac{\pi}{6}$.

$y = \frac{1}{2}$.

$x = \frac{\sqrt{3}}{2}$.

2) a) i) $\angle ASK = 90^\circ$ (angle in semi circle)

$\angle ADB = 90^\circ$ (corr. $\angle$'s $CB \parallel SK$)

$\therefore AS \perp BC$

ii) let $\angle BAK = \alpha$

$\angle ABK = 90^\circ$ (angle in semi circle)

$\angle BKA = 90 - \alpha$ ($\angle$ sum of $\triangle$)

$\angle ACB = 90 - \alpha$ ($\angle$ in same segment)

$\angle ADC = 90^\circ$ ($AS \perp BC$)

$\angle SAC = \alpha$ ($\angle$ sum of $\triangle$)

b) i) $LHS = u^2 + u + 1 + \frac{1}{u-1}$

$$= \frac{(u-1)(u^2+u+1) + 1}{u-1}$$

$$= \frac{u^3 - 1 + 1}{u-1}$$

$$= \frac{u^3}{u-1}$$

$= RHS$

ii) $$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}}$$

$x = u^6$ $\quad$ $u = x^{\frac{1}{6}}$

$\frac{dx}{du} = 6u^5$

$dx = 6u^5\, du$

$$= \int \frac{6u^5\, du}{u^3 - u^2}$$

$$= \int \frac{6u^3}{u-1}\, du$$

$$= 6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right) du$$

$$= 6\left[\frac{u^3}{3} + \frac{u^2}{2} + u + \ln(u-1)\right] + c$$

$$= 6\left[\frac{\sqrt{x}}{3} + \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} + \ln\left(\sqrt[6]{x} - 1\right)\right] + C$$

c) $V = \pi \int_a^b y^2 \, dx$

$V = \pi \int_0^2 \frac{9}{x^2+4} \, dx$

$V = 9\pi \int_0^2 \frac{dx}{x^2+4}$

$V = 9\pi \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_0^2$

$V = 9\pi \left[\frac{1}{2} \tan^{-1} \frac{2}{2} - \cancel{\frac{1}{2} \tan^{-1} \frac{0}{2}} \right]$

$V = 9\pi \left[\frac{1}{2} \times \frac{\pi}{4} \right]$

$V = \frac{9\pi^2}{8}$ units^3

d) i) $f(x) = 2 - \frac{1}{2e^{x^3}}$

$= 2 - \frac{1}{2} e^{-x^3}$

$f'(x) = \frac{3x^2}{2} e^{-x^3}$

$f'(x) + 3x^2 f(x) = \frac{3x^2}{2} e^{-x^3} + 3x^2 \left(2 - \frac{1}{2} e^{-x^3}\right)$

$= \cancel{\frac{3x^2}{2} e^{-x^3}} + 6x^2 - \cancel{\frac{3x^2}{2} e^{-x^3}}$

$= 6x^2$

ii) $f(-2) = 2 - \frac{1}{2} e^{-(-2)^3}$

$= -1488.48 \ldots$

< 0

$f(-1) = 2 - \frac{1}{2} e^{-(-1)^3}$

$= 0.640859 \ldots$

> 0

since the continuous function $f(x)$ has a change in sign at least one root lies between $x=-2$ & $x=-1$.

iii) $a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}$

$f(-1) = 0.640859085$

$f'(-1) = \frac{3}{2}(-1)^2 e^{-(-1)^3}$

$= 4.077422743$

$a_1 = -1 - \frac{0.640859085}{4.077422743}$

$a_1 = -1.157$ (3 dec. places)

e) $\int_{-3}^{0} \frac{x^2}{\sqrt{1-x}}\,dx$

$u = 1-x$ $\qquad x = 1-u$

$\frac{du}{dx} = -1$

$dx = -du$

when $x=0$, $u=1$ $\qquad x=-3$, $u=4$

$= \int_4^1 \frac{(1-u)^2}{\sqrt{u}} \cdot -du$

$= \int_1^4 \frac{1-2u+u^2}{\sqrt{u}}\,du$

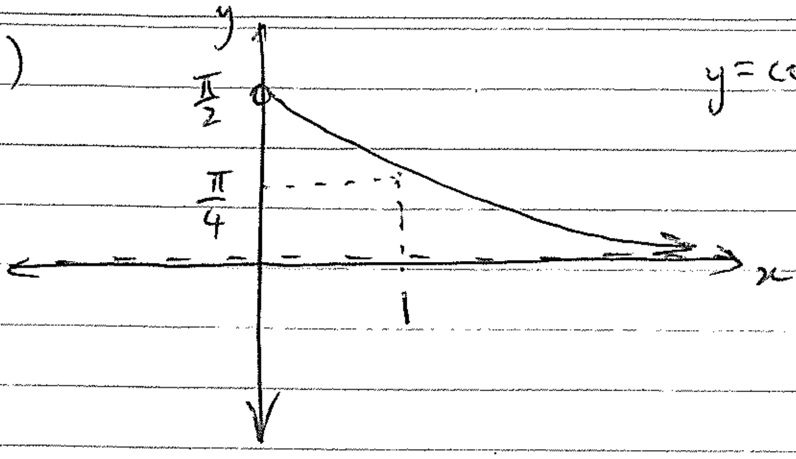
$= \int_1^4 \left(u^{-\frac{1}{2}} - 2u^{\frac{1}{2}} + u^{\frac{3}{2}}\right) du$

$= \left[2u^{\frac{1}{2}} - \frac{4}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}}\right]_1^4$

$= \left[2(4)^{\frac{1}{2}} - \frac{4}{3}(4)^{\frac{3}{2}} + \frac{2}{5}(4)^{\frac{5}{2}} - \left(2 - \frac{4}{3} + \frac{2}{5}\right)\right]$

$= \frac{76}{15}$

f) i)



$y = \cot^{-1} x \rightarrow x = \cot y$

$0 < y < \frac{\pi}{2}$

ii) $y = \cot^{-1} x$

$x = \cot y$

$\frac{dx}{dy} = -\operatorname{cosec}^2 y$

$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y}$

$= \frac{-1}{1+\cot^2 y}$

$= \frac{-1}{1+x^2}$

OR

$x = \frac{1}{\tan y}$

$\tan y = \frac{1}{x}$

$y = \tan^{-1}\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot -\frac{1}{x^2}$

$= \frac{-1}{1+x^2}$

iii) $f(x) = \tan^{-1} x + \cot^{-1} x$

$f'(x) = \frac{1}{1+x^2} + -\frac{1}{1+x^2}$

$= 0$

$\therefore f(x)$ is a constant function

iv) $f(1) = \tan^{-1} 1 + \cot^{-1} 1$

$= \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$ $\therefore f(x) = \frac{\pi}{2}$

Question 3

(a) $I = \int_0^\pi x\, f(\sin x)\,dx$

Let $u = \pi - x$

$\therefore du = -dx$

When $x = 0$, $u = \pi$

$x = \pi$, $u = 0$

$\therefore I = \int_\pi^0 (\pi - u) f(\sin(\pi - u))(-du)$

$= \int_0^\pi \pi f(\sin u)\,du - \int_0^\pi u f(\sin u)\,du$

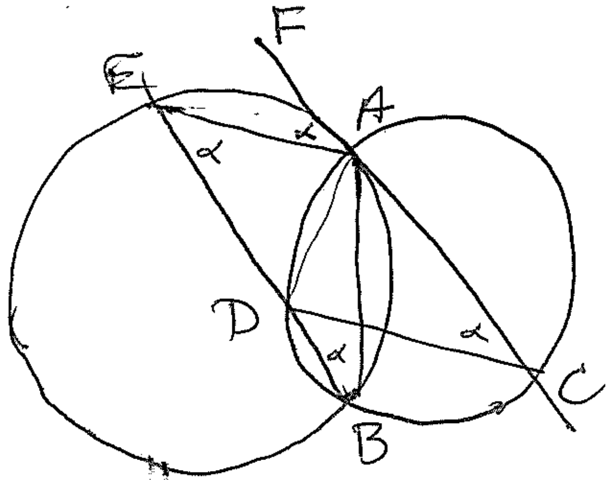
$= \int_0^\pi \pi f(\sin x)\,dx - \int_0^\pi x f(\sin x)\,dx$ (*)

$\therefore 2I = \int_0^\pi \pi f(\sin x)\,dx$

[3]

(*) The variable name is a dummy — swap u for x.

(b)



Join AB

Let $\angle FAE = \alpha$

$\therefore \angle AEB = \alpha$ (alternate ∠s, $AE \parallel CD$)

$\angle ABE = \alpha$ (alternate segment)

$\angle ACD = \alpha$ (standing on same chord)

ie $\angle FAE = \angle ACD$ and so

$AE \parallel DC$ (Corresp. angles)

$\therefore$ EACD is a parallelogram (a pair of sides equal & parallel)

$\therefore EA = DC$ (opp sides of parallelogram equal)

QED

[3]

(c) p(n): Aim to prove

$19 \mid 5\times 2^{3n-2} + 3^{3n-1}$

p(1): Test when $n = 1$

$5\times 2^{3-2} + 3^{3-1}$

$= 19$ [1]

p(k): Assume true for $n = k$

ie $5\times 2^{3k-2} + 3^{3k-1} = 19R$

p(k+1): RTP that $p(k) \rightarrow p(k+1)$

$5\times 2^{3k+1} + 3^{3k+2}$

$= 40\times 2^{3k-2} + 27(19R - 5\times 2^{3k-2})$

$= 40\times 2^{3k-2} + 27\times 19R - 135\times 2^{3k-2}$

$= 27\times 19R - 95\times 2^{3k-2}$

$= 19(27R - 5\times 2^{3k-2})$ [2]

$\therefore p(k) \rightarrow p(k+1)$.

Thus p(n) is true for $n \geq 1$.

(d) $f(x) = \dfrac{e^x - e^{-x}}{2}$

(i) Invert by exchanging x and y.

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2xe^y = e^{2y} - 1 \quad (\text{mpy by } e^y)$$

$$e^{2y} - 2xe^y - 1 = 0 \quad (\text{quadratic})$$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

Take logs, drop spurious -ve

$$y = \ln\left(x + \sqrt{x^2+1}\right) \quad [3]$$

(ii) $5 = \dfrac{e^x - e^{-x}}{2}$

So $e^{2x} - 10e^x - 1 = 0$

$$e^x = \frac{10 + \sqrt{100+4}}{2}$$

$$= 10.099\ldots$$

$$x \doteqdot 2.31 \quad [1]$$

(iii) $\dfrac{d}{dx}\left(\ln\left(x + \sqrt{x^2+1}\right)\right)$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\sqrt{x^2+1}} \quad [2]$$

(e) (i) $\tan\phi = \dfrac{0.4}{x}$

$$\therefore \phi = \tan^{-1}\frac{0.4}{x} \quad [1]$$

(ii) $\theta + \phi = \tan^{-1}\left(\dfrac{2.5}{x}\right)$

$$\therefore \theta = \tan^{-1}\left(\frac{2.5}{x}\right) - \tan^{-1}\left(\frac{0.4}{x}\right) \quad [2]$$

(iii) $\dfrac{d\theta}{dx} = \dfrac{-2.5}{x^2+6.25} + \dfrac{0.4}{x^2+0.16}$

$$= \frac{-2.5(x^2+0.16) + 0.4(x^2+6.25)}{(x^2+6.25)(x^2+0.16)}$$

$\dfrac{d\theta}{dx} = 0$ when

$$-2.5(x^2+0.16) + 0.4(x^2+6.25) = 0$$

$$2.1(1 - x^2) = 0$$

$$1 - x^2 = 0$$

$$\therefore x = \pm 1$$

But $x > 0$

$$\therefore x = 1 \quad [3]$$